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Repeated Integral Inequalities

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Generalizations of the classical Gronwall inequality when the integral inequality involves a repeated integral are presented. The corresponding discrete versions are deduced from the continuous results by employing a further Gronwall inequality. An application is given.

1. Introduction

IN 1919 GRONWALL proved the following result.

LEMMA 1.1 *Let the function x be continuous and non-negative on the interval* $[0, T]$. *If*

$$x(t) \leqslant a+b\int_0^t x(s)\,ds, \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

where a, b are non-negative constants, then

$$x(t) \leqslant a\,e^{bt}, \quad 0 \leqslant t \leqslant T. \tag{1.2}$$

Gronwall inequalities have become a generic term for inequalities of this type which give explicit bounds for a function which satisfies a Volterra integral inequality. The applications of such inequalities are numerous in the establishing of existence and uniqueness of solutions of integral and ordinary differential equations, and in the perturbation and stability analysis of ordinary differential equations. Some of these applications appear in Walter (1970), Lakshmikantham & Leela (1969) and Bellman (1953).

Recurrent inequalities involving sequences of real numbers, which may be considered to be discrete Gronwall inequalities, have been widely used in the analysis of finite difference equations. The book by Henrici (1962) provides an elementary introduction to the application of such results to the numerical solution of ordinary differential equations. The following lemma, which is encountered frequently in numerical analysis, may be regarded as the discrete analogue of Lemma 1.1.

LEMMA 1.2 *Let* x_i, $i = 0, 1, \ldots, N$ *be a sequence of non-negative real numbers satisfying*

$$x_0 \leqslant \delta, \quad x_i \leqslant \delta+Mh\sum_{j=0}^{i-1} x_j, \quad i = 1, 2, \ldots, N, \tag{1.3}$$

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where δ, M are non-negative constants with M bounded independently of $h(=T/N)$, then

$$x_i \leqslant \delta \exp(Mih), \quad i = 0, 1, \ldots, N. \tag{1.4}$$

A reasonably comprehensive account of Gronwall inequalities, which includes a discussion of both continuous and discrete inequalities, has been given by Beesack (1975).

The purpose of this note is to present linear generalizations of Lemma 1.1 when the integral inequality involves a repeated integral, and to derive the corresponding discrete results from the continuous results.

The principal motivation behind this work is the derivation of discrete Gronwall inequalities which will facilitate convergence proofs of discretization methods for integro-differential equations with continuous or weakly singular kernels.

In our analysis we will require the following linear generalization of Gronwall's inequality due to Chu & Metcalf (1967).

LEMMA 1.3 *Let the functions x, ϕ be continuous on the interval $[0, T]$. Let $k(t, s)$ be continuous and non-negative on the triangle $0 \leqslant s \leqslant t \leqslant T$. If*

$$x(t) \leqslant \phi(t) + \int_0^t k(t, s)x(s)\, ds, \quad 0 \leqslant t \leqslant T, \tag{1.5}$$

then

$$x(t) \leqslant y(t), \quad 0 \leqslant t \leqslant T, \tag{1.6}$$

where y is the unique continuous solution of the Volterra integral equation

$$y(t) = \phi(t) + \int_0^t k(t, s)y(s)\, ds, \quad 0 \leqslant t \leqslant T. \tag{1.7}$$

This result (and all subsequent results) remains valid if x, ϕ are bounded and continuous almost everywhere on $[0, T]$ (the solution y of (1.7) will also be bounded and continuous almost everywhere). This enables us to include piecewise continuous functions.

2. Deducing Discrete Results from the Corresponding Continuous Results

We first present a discrete Gronwall inequality which will allow us to deduce discrete results from the corresponding continuous ones.

THEOREM 2.1 *Let x_i, $i = 0, 1, \ldots, N$ be a sequence of non-negative real numbers satisfying*

$$x_i \leqslant \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j, \quad i = 0, 1, \ldots, N, \tag{2.1}$$

where ϕ_i, $i = 0, 1, \ldots, N$ is a sequence of non-negative finite real numbers, and $0 \leqslant k_{ij} \leqslant M$, $0 \leqslant j < i \leqslant N$, for some M bounded independently of $h(=T/N)$.

If there exists a continuous, non-negative function $k(t, s)$ defined on the triangle $0 \leqslant s \leqslant t \leqslant T$ such that

$$k(t, s) \geqslant k_{ij},$$

for

$$ih \leqslant t < (i+1)h, \quad jh \leqslant s < (j+1)h, \quad 0 \leqslant j < i < N,$$

and

$$k(t, s) \geqslant k_{Nj},$$

for

$$t = Nh, \quad jh \leqslant s < (j+1h), \quad 0 \leqslant j < N,$$

then

$$x_i \leqslant y(ih), \quad i = 0, 1, \ldots, N, \tag{2.2}$$

where y is the unique solution of the integral equation

$$y(t) = \phi(t) + \int_0^t k(t, s) y(s)\, ds, \quad 0 \leqslant t \leqslant T,$$

and $\phi(t)$ *is the step function defined on* $[0, T]$ *by*

$$\phi(t) = \begin{cases} \phi_i, & ih \leqslant t < (i+1)h, \quad i = 0, 1, \ldots, N-1 \\ \phi_N, & t = Nh = T. \end{cases}$$

(Here and elsewhere we assume that

$$\sum_{j \in \theta} D_j = 0 \quad \text{and} \quad \prod_{j \in \theta} D_j = 1$$

if θ *is the empty set; thus* $x_0 \leqslant \phi_0$ *is assumed in* (2.1).)

Proof. Since ϕ_i, $i = 0, 1, \ldots, N$ is a sequence of finite real numbers and $0 \leqslant k_{ij} \leqslant M$ for some constant M,

$$x_i \leqslant \Phi + Mh \sum_{j=0}^{i-1} x_j, \quad i = 0, 1, \ldots, N,$$

where

$$\Phi = \max_i \phi_i.$$

Applying Lemma 1.2,

$$x_i \leqslant \Phi \exp(Mih), \quad i = 0, 1, \ldots, N,$$

and consequently x_i, $i = 0, 1, \ldots, N$, is bounded.

We may now define a step function $x(t)$ on $[0, T]$ as follows:

$$x(t) = \begin{cases} x_i, & ih \leqslant t < (i+1)h, \quad i = 0, 1, \ldots, N-1 \\ x_N, & t = Nh = T. \end{cases}$$

For any $t \in [0, T)$ there exists a unique i, $0 \leqslant i \leqslant N-1$, such that $ih \leqslant t < (i+1)h$. With this t,

$$x(t) = x_i \leqslant \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j$$

and

$$\begin{aligned} \phi(t) + \int_0^t k(t, s) x(s)\, ds &= \phi_i + \sum_{j=0}^{i-1} \int_{jh}^{(j+1)h} k(t, s) x(s)\, ds + \int_{ih}^t k(t, s) x(s)\, ds \\ &= \phi_i + \sum_{j=0}^{i-1} x_j \int_{jh}^{(j+1)h} k(t, s)\, ds + x_i \int_{ih}^t k(t, s)\, ds \\ &\geqslant \phi_i + \sum_{j=0}^{i-1} k_{ij} x_j \geqslant x_i = x(t). \end{aligned}$$

Similarly, if $t = T$,

$$x(t) = x_N \leqslant \phi_N + h \sum_{j=0}^{N-1} k_{Nj} x_j$$

and

$$\phi(t) + \int_0^t k(t, s)x(s)\,ds \geqslant x_N = x(t).$$

Therefore, for every $t \in [0, T]$

$$x(t) \leqslant \phi(t) + \int_0^t k(t, s)x(s)\,ds.$$

Invoking Lemma 1.3 (with x, ϕ bounded and continuous almost everywhere on $[0, T]$)

$$x(t) \leqslant y(t), \quad 0 \leqslant t \leqslant T,$$

where

$$y(t) = \phi(t) + \int_0^t k(t, s)y(s)\,ds, \quad 0 \leqslant t \leqslant T.$$

Letting $t = ih$ we conclude that

$$x_i \leqslant y(ih), \quad i = 0, 1, \ldots, N. \tag{2.2}$$

3. A Linear Generalization of Gronwall's Inequality

We are interested in integral equations of the form

$$y(t) = \phi(t) + \int_0^t \int_0^{t_m} \ldots \int_0^{t_1} \frac{k(t, s, y(s))}{(t_1 - s)^\alpha}\,ds\,dt_1 \ldots dt_m, \tag{3.1}$$

where $\alpha < 1$ and m is a natural number. The functions ϕ, k are assumed to satisfy suitable continuity conditions, and $k(t, s, y)$ is assumed Lipschitz continuous with respect to y.

Integral equations of this form or, more precisely, the corresponding integral inequalities, may be used to demonstrate uniqueness and boundedness of the solution of integro-differential equations with an Abel's type singularity

$$y^{(m)}(t) = \psi(t) + \int_0^t \frac{k(t, s, y(s))}{(t-s)^\alpha}\,ds, \quad 0 \leqslant t \leqslant T, \tag{3.2}$$

where $\alpha < 1$ and $m \geqslant 1$. (Here $y^{(m)}(t)$ denotes the mth derivative of y with respect to t.)

LEMMA 3.1 *Let the function x be continuous and non-negative on the interval $[0, T]$. If*

$$x(t) \leqslant \phi(t) + M \int_0^t \int_0^{t_m} \ldots \int_0^{t_1} \frac{x(s)}{(t_1 - s)^\alpha}\,ds\,dt_1 \ldots dt_m, \quad 0 \leqslant t \leqslant T, \tag{3.3}$$

where $\alpha < 1$, $m \geqslant 1$, $M > 0$ is constant, and $\phi(t)$ is a non-negative, non-decreasing continuous function in t, $0 \leqslant t \leqslant T$, then

$$x(t) \leqslant \phi(t) E_{1-(\alpha-m)}(M\Gamma(1-\alpha)t^{1-(\alpha-m)}), \quad 0 \leqslant t \leqslant T, \tag{3.4}$$

where $E_{1-\beta}(z)$ is the Mittag–Leffler function defined for any β by

$$E_{1-\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\beta)+1)},$$

and $\Gamma(a)$ is the Gamma function defined for $\operatorname{Re} a > 0$ *by*

$$\Gamma(a) = \int_0^{\infty} w^{a-1} e^{-w}\, dw.$$

The exponential function, which is obtained when $\beta = 0$, is a special case of the Mittag–Leffler function.

The Mittag–Leffler function has been studied in some detail in the literature; for references see Erdèlyi (1955, ch. 18).

Proof. For $m \geqslant 1$, by interchanging the order of integration,

$$\int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{x(s)}{(t_1-s)^{\alpha}}\, ds\, dt_1 \ldots dt_m = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \int_0^t (t-s)^{m-\alpha} x(s)\, ds.$$

Consequently, the inequality (3.3) is equivalent to

$$x(t) \leqslant \phi(t) + \int_0^t k(t,s) x(s)\, ds,$$

where the kernel $k(t, s)$ given by

$$k(t,s) = \frac{M\Gamma(1-\alpha)(t-s)^{m-\alpha}}{\Gamma(1-\alpha+m)}, \quad 0 \leqslant s \leqslant t \leqslant T, \tag{3.5}$$

$\alpha < 1$, $m \geqslant 1$, is continuous and non-negative.

Invoking Lemma 1.3,

$$x(t) \leqslant y(t),$$

where $y(t)$ is the solution of the integral equation (1.7) with kernel (3.5).

The (unique) solution of (1.7) is given by

$$y(t) = \phi(t) + \int_0^t \Gamma(t,s)\phi(s)\, ds, \quad 0 \leqslant t \leqslant T,$$

where

$$\Gamma(t,s) = \sum_{n=1}^{\infty} k^{(n)}(t,s), \quad 0 \leqslant s \leqslant t \leqslant T,$$

is the resolvent kernel of $k(t, s)$ and $k^{(n)}(t, s)$ $(n = 1, 2, \ldots)$ are the iterated kernels of $k(t, s)$ defined by

$$k^{(1)}(t,s) = k(t,s)$$
$$k^{(n)}(t,s) = \int_s^t k(t,u) k^{(n-1)}(u,s)\, du, \quad n \geqslant 2.$$

Using mathematical induction it can be shown that the iterated kernels satisfy

$$k^{(n)}(t,s) = \frac{M^n \Gamma(1-\alpha)^n (t-s)^{n(1-\alpha+m)-1}}{\Gamma(n(1-\alpha+m))}, \quad n = 1, 2, \ldots. \tag{3.6}$$

Hence we obtain

$$x(t) \leqslant \phi(t) + \sum_{n=1}^{\infty} \frac{(M\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha+m))} \int_0^t (t-s)^{n(1-\alpha+m)-1} \phi(s)\, ds$$
$$\leqslant \phi(t) E_{1-(\alpha-m)}(M\Gamma(1-\alpha) t^{1-(\alpha-m)}), \quad 0 \leqslant t \leqslant T. \tag{3.4}$$

Special Case. If $\alpha = 0$, $m = 1$ (3.4) reduces to

$$x(t) \leqslant \phi(t) \cosh (M^{\frac{1}{2}} t).$$

Note that in the case $\phi(t) = \phi$, $0 \leqslant t \leqslant T$, (3.4) is the best possible result since equality in (3.3) gives equality in (3.4). For a more general $\phi(t)$, the best possible result is given by

$$x(t) \leqslant \frac{d}{dt} \int_0^t E_{1-(\alpha-m)}(M\Gamma(1-\alpha)(t-s)^{1-(\alpha-m)}) \phi(s)\, ds, \tag{3.4'}$$

where the right-hand side of (3.4′) is the solution of the integral equation (1.7) with kernel (3.5).

We also remark that if $\alpha \leqslant 0$, Lemma 3.1 remains valid if $m = 0$, that is, if (3.3) involves a single, rather than repeated, integral and in this case Lemma 3.1 is an example of Lemma 1.3. If $0 < \alpha < 1$ and $m = 0$ the kernel $k(t, s) = M/(t-s)^\alpha$ is weakly singular; Gronwall inequalities where the kernel of the associated integral equation is weakly singular can be found in Dixon & McKee (1984).

4. A Discrete Gronwall Inequality

In this section we present the discrete analogue of Lemma 3.1.

LEMMA 4.1 *Let x_i, $i = 0, 1, \ldots, N$ be a sequence of non-negative real numbers satisfying*

$$x_i \leqslant \phi_i, \quad i = 0, 1, \ldots, m,$$
$$x_i \leqslant \phi_i + Mh^{1-(\alpha-m)} \sum_{i_m=0}^{i-1} \sum_{i_{m-1}=0}^{i_m-1} \cdots \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha}, \quad i = m, m+1, \ldots, N, \tag{4.1}$$

where $\alpha < 1$, $m \geqslant 1$, $M > 0$ is bounded independently of h, and ϕ_i, $i = 0, 1, \ldots, N$ is a non-decreasing sequence of non-negative finite real numbers, then

$$x_i \leqslant \phi_i E_{1-(\alpha-m)}(M\Gamma(1-\alpha)(ih)^{1-(\alpha-m)}), \quad i = 0, 1, \ldots, N. \tag{4.2}$$

Proof. We first proceed by mathematical induction to show that for $m \geqslant 1$

$$\sum_{i_m=0}^{i-1} \sum_{i_{m-1}=0}^{i_m-1} \cdots \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} \leqslant \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \sum_{j=0}^{i-1} (i-j-1)^{m-\alpha} x_j. \tag{4.3}$$

Consider the case $m = 1$. Interchanging the order of summation

$$\sum_{i_1=0}^{i-1} \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} = \sum_{j=0}^{i-2} \left(\sum_{i_1=j+1}^{i-1} \frac{1}{(i_1-j)^\alpha} \right) x_j.$$

We bound the inner summation on the right-hand side by an integral as follows:

$$\sum_{i_1=j+1}^{i-1} \frac{1}{(i_1-j)^\alpha} \leqslant \sum_{i_1=j+1}^{i-1} \int_{i_1-1}^{i_1} \frac{1}{(s-j)^\alpha}\, ds = \frac{(i-j-1)^{1-\alpha}}{1-\alpha}.$$

Therefore, since $x_i \geqslant 0$, $0 \leqslant i \leqslant N$,

$$\sum_{i_1=0}^{i-1} \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} \leqslant \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (i-j-1)^{1-\alpha} x_j$$

and (4.3) holds when $m = 1$.

Using similar arguments it can also be shown that if (4.3) is assumed to hold for $m = n$, then it also holds for $n+1$, and hence the induction is complete. Consequently (4.1) implies

$$x_i \leqslant \phi_i + \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} h^{1-(\alpha-m)} \sum_{j=0}^{i-1} (i-(j+1))^{m-\alpha} x_j, \quad i = 0, 1, \ldots, N.$$

This is of the form

$$x_i \leqslant \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j,$$

where

$$0 \leqslant k_{ij} \leqslant \frac{M\Gamma(1-\alpha)T^{m-\alpha}}{\Gamma(1-\alpha+m)} = \hat{M},\ 0 \leqslant j < i \leqslant N,$$

where $\hat{M}$ is bounded independently of h. Furthermore,

$$k_{ij} = \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (h(i-(j+1)))^{m-\alpha} \leqslant \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (t-s)^{m-\alpha} = k(t, s)$$

for

$$ih \leqslant t < (i+1)h, \quad jh \leqslant s < (j+1)h, \quad 0 \leqslant j < i < N,$$

and

$$k_{Nj} = \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (h(N-(j+1)))^{m-\alpha} \leqslant \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (T-s)^{m-\alpha} = k(T, s)$$

for

$$t = Nh, \quad jh \leqslant s < (j+1)h, \quad 0 \leqslant j < N.$$

Applying Theorem 2.1 and employing Lemma 3.1 yields the required result.

Special Case. If $\alpha = 0$, $m = 1$ (4.2) reduces to

$$x_i \leqslant \phi_i \cosh (M^{\frac{1}{2}}(ih)).$$

We again remark that Lemma 4.1 remains valid for $m = 0$ if $\alpha \leqslant 0$. For the case $m = 0$ and $0 < \alpha < 1$ we again refer to Dixon & McKee.

5. An Application

Consider the integro-differential equation

$$y'(t) = F(t, y(t), \psi(t)), y(0) \text{ given}, \quad 0 \leqslant t \leqslant T, \tag{5.1}$$

where

$$\psi(t) = \int_0^t \frac{k(t, s, y(s))\, ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1. \tag{5.2}$$

In the subsequent discussion it will be assumed that F, k are sufficiently smooth to guarantee the existence of a unique solution y which has a bounded second derivative.

We note that recently Brunner (1982) has studied collocation methods for solving Equation (5.1) where $F(t, y, \psi)$ is linear.

Applying an l-step linear multistep method to the differential part of (5.1) gives

$$\frac{1}{h}\sum_{j=0}^{l} a_j y_{i-j} = \sum_{j=0}^{l} b_j F(t_{i-j}, y_{i-j}, z_{i-j}), \tag{5.3}$$

where y_i denotes an approximation to $y(t_i)$, $t_i = ih$, $0 \leqslant i \leqslant N$, $Nh = T$, and z_i denotes an approximation to $\psi(t_i)$.

To illustrate an application of Lemma 4.1 a simple product integration method is used to approximate $\psi(t_i)$,

$$z_i = h\sum_{j=0}^{i-1} w_{ij} k(t_i, t_j, y_j), \tag{5.4}$$

where

$$w_{ij} = \frac{1}{h}\int_{t_j}^{t_{j+1}} \frac{ds}{(t_i - s)^{\alpha}},$$

and Euler's method is chosen as the linear multistep method (5.3).

This permits us to define the discrete algorithm

$$\Phi^h(y^h) = 0, \quad \Phi^h : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1},$$

where

$$[\Phi^h(y^h)]_i = \begin{cases} y_i - \tilde{y}_i, & i = 0, 1 \\ \left.\begin{aligned} &\frac{y_i - y_{i-1}}{h} - F(t_{i-1}, y_{i-1}, z_{i-1}), \\ &z_{i-1} = h\sum_{j=0}^{i-2} w_{i-1,j} k(t_{i-1}, t_j, y_j). \end{aligned}\right\} & i = 2, 3, \ldots, N. \end{cases} \tag{5.5}$$

It is assumed that the required starting values $\tilde{y}_0$, $\tilde{y}_1$ are accurate of order one.

We observe that since $i - j \geqslant 1$,

$$|w_{ij}| = w_{ij} = \frac{1}{h^{\alpha}(i-j)^{\alpha}}\int_0^1 \frac{du}{(1 - u/(i-j))^{\alpha}} \leqslant \frac{h^{-\alpha}}{(1-\alpha)(i-j)^{\alpha}}. \tag{5.6}$$

Using the bound (5.6) it can be shown that the discretization (5.5) is consistent of order at least one, that is, there exist constants C_i, independent of h, such that

$$|\theta_i^h| := \left|\frac{y(t_i) - y(t_{i-1})}{h} - F(t_{i-1}, y(t_{i-1}), z(t_{i-1}))\right| \leqslant C_i h, \tag{5.7}$$

where

$$z(t_i) = h\sum_{j=0}^{i-1} w_{ij} k(t_i, t_j, y(t_j)).$$

To demonstrate the convergence of the discretization (5.5), we shall consider the special case

$$F(t, y(t), \psi(t)) = f(t) + \psi(t).$$

The more general case will follow in a similar manner but notationally it is more complicated.

The error $y(t_i)-y_i$ satisfies

$$(y(t_i)-y_i)-(y(t_{i-1})-y_{i-1}) = h(z(t_{i-1})-z_{i-1})+h\theta_i^h.$$

Summing over $i = 2$ to k,

$$y(t_k)-y_k = h\sum_{i=1}^{k-1}(z(t_i)-z_i)+h\sum_{i=2}^{k}\theta_i^h+y(t_1)-\tilde{y}_1.$$

Using the bound (5.6) and consistency, and assuming that the starting values are accurate of order one, and that $k(t, s, y)$ is Lipschitz continuous with respect to y with Lipschitz constant L, $x_i = |y(t_i)-y_i|$ satisfies

$$x_0 \leqslant Ch$$

$$x_i \leqslant Ch+\frac{Lh^{2-\alpha}}{1-\alpha}\sum_{k=0}^{i-1}\sum_{j=0}^{k-1}\frac{x_j}{(k-j)^\alpha}, \quad i = 1, 2, \ldots, N,$$

for some constant C.

Convergence of order at least one now follows by applying Lemma 4.1.

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